Proof in axiomatic set theory (ZFC): ~Finite(A) ==> $A \approx (A \cup \{A\})$

Lemma1: ~finite(A) ==> $\exists B (B \subseteq A \ \wedge \ B \approx \mathbb{N})$.
Proof:
we have $A \neq \{\}$, hence there exists $b_1 \in A$.
put $B_1 = \{\}$ and for n>1: $B_n = B_{n-1} \cup \{b_{n-1}\}$,
we have $A\text{-}B_n \neq \{\}$: by contradiction $A\text{-}B_n = \{\}$, hence $A \subseteq B_n$, hence finite(A).
We put $B = \cup B_n$ and get $B \approx \mathbb{N}$ by definition.
qed

Lemma2: Let A be an arbitrary set and B countably infinite ==> $\exists g \in \text{bij}(B, B\cup\{A\})$.
Proof:
B has the representation $B = \{b_1, b_2, b_3 .....\}$
We define function g as follows:
$g(b_1) = A$,
$g(b_2) = g(b_1)$
....
$g(b_n) = g(b_{n-1})$, n>1.
qed

with the help of lemma1 and lemma2 we prove
lemma3: ~Finite(A) ==> $A \approx (A \cup \{A\})$.
Proof:
Let A be a set with ~Finite(A).
By virtue of lemma1 there is a subset $B \subseteq A$ and $B \approx \mathbb{N}$.
By virtue of lemma2 there is a mapping $g \in \text{bij}(B, B\cup\{A\})$.
Now we define the mapping f: A --> $A \cup \{A\}$ as follows:
$x \in B$: $f(x)=g(x)$
$x \notin B$: $f(x)=x$.
f maps B to $B\cup\{A\}$ and A-B to A-B bijectively, i.e. $f \in \text{bij}(A, A\cup\{A\})$.
qed

Translated to Isabelle:

```
Theory First imports ZF.OrderType ZF ZF.Finite ZF.Nat ZF.Sum ZF.AC

begin

definition eqpoll (infix "≅" 70) where "A≅B ≡ ∃f. f∈bij(A,B)"

definition "FiniteSet(A) ≡ ∃n∈nat. A ≅ n"


lemma ad1: "∀A.∃B.(~finite(A) ==> (B ⊆ A  ∧  B ≅ ℕ))" by auto
lemma ad2: "∀A.∀B.∃g.(B ≅ ℕ ==>  g ∈ bij(B, B∪{A}))"  by auto
lemma ad3: "∀A.(~Finite(A) ==> A ≅ (A ∪ {A}))"
  apply (ad1)
  apply (ad2)
  done

end
```

in First.thy following error messages occur in Isabelle:
- Inner lexical error
- Inner syntax error: unexpected end of input