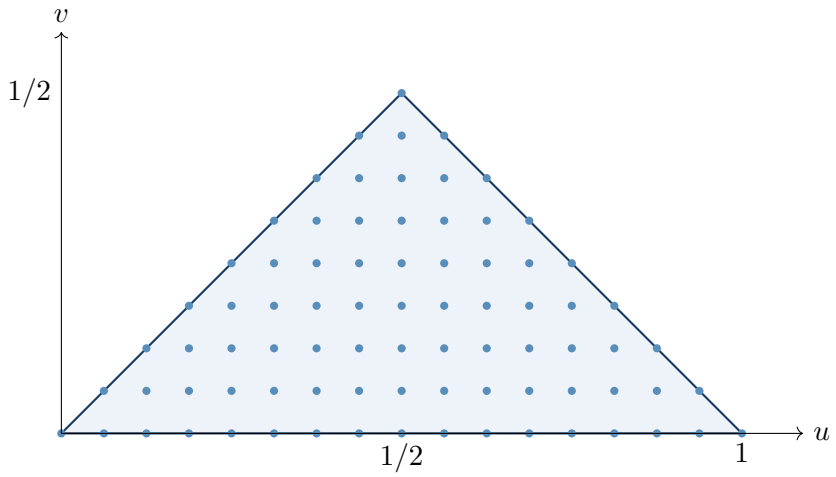
Emergent Geometry from Algebraic Structures

Orthogonal Polynomials, Discrete Geometry
and Global Curvature Balance



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Introduction

The origin of geometric spaces and microscopic structures is a fundamental question at the intersection of theoretical physics and mathematics. Of particular interest is the relationship between algebraic laws and geometric order. This study considers how geometric structures can arise from an algebraic system in which spatial properties are not specified in advance.

The mathematical starting point is a family of orthogonal polynomials whose multiplication rules produce an ordered structure of interrelated coefficients. These coefficients form a three-dimensionally indexed lattice whose shape and internal relations are determined by algebraic laws. This makes it possible to examine geometric properties without imposing them as spatial assumptions at the outset.

The lattice's symmetries and local relations then lead to a discrete curvature quantity, a precisely bounded geometric limiting region, and an exact global curvature balance. Together, these properties describe how algebraic multiplication rules give rise to geometric order with well-defined boundaries and symmetries.

1 Orthogonal Polynomials and Recurrence

The mathematical starting point is a family of orthogonal polynomials whose structure allows geometric order to be developed from algebraic multiplication rules. Polynomials are functions built from powers of a variable. Their products can be expanded in such a way that the resulting coefficients satisfy definite relations. These relations will provide the basis for the three-dimensional lattice and its geometric properties.

The polynomial family is defined by a three-term recurrence. With $P_0(x) = 1$ and $P_1(x) = x$, we have

$$xP_n(x) = P_{n+1}(x) + b_n P_{n-1}(x), \qquad n \geq 1,$$
$$b_n = \frac{n(n+2)(n+1)^2}{4(2n+1)(2n+3)} > 0$$

The recurrence determines each successive polynomial from the preceding two. All polynomials are monic, meaning that the coefficient of the highest-degree term is one. Since $b_n > 0$ for $n \geq 1$, Favard's theorem guarantees orthogonality with respect to a suitable positive measure.

The first polynomials obtained from this rule are:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{5}, \quad P_3(x) = x^3 - \frac{5}{7}x, \quad P_4(x) = x^4 - \frac{5}{3}x^2 + \frac{4}{21}.$$

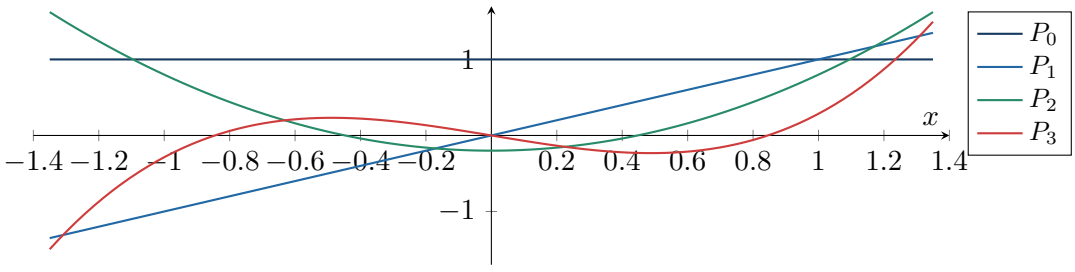
The polynomials also satisfy a parity relation:

$$P_n(-x) = (-1)^n P_n(x).$$

Even-indexed polynomials contain only even powers, while odd-indexed polynomials contain only odd powers. This restricts the degrees that can occur in subsequent product expansions. The growth of the recurrence coefficients at large indices is also determined:

$$b_n = \frac{n^2}{16}\left(1 + O\left(\frac{1}{n}\right)\right), \qquad \lim_{n\to\infty} \frac{16 b_n}{n^2} = 1.$$

Thus, the coefficients grow quadratically in the asymptotic regime. Recurrence, orthogonality, and parity together establish the algebraic basis from whose product expansions the lattice values are obtained.



2 Product Expansions and Coefficients

Multiplying two polynomials produces another polynomial. For the geometric construction, this product is expressed again in terms of the original polynomial family. The resulting numbers specify the weight of each basis polynomial in the expansion. Since $P_0, \ldots, P_N$ form a basis by virtue of their distinct degrees and monicity, this decomposition is unique. Orthogonality and parity restrict the possible terms:

$$P_m(x)P_n(x) = \sum_{j=0}^{\min(m,n)} F_j(m,n)P_{m+n-2j}(x).$$

The numbers $F_j(m,n)$ are called linearization coefficients. They are the data underlying the construction: each multiplication produces a finite, uniquely determined list of values. To calculate them, the product is expanded and then rewritten in the basis P_k, proceeding from the highest degree downward.

2.1 Example: Expansion of P_2^2

For $P_2(x) = x^2 - \frac{1}{5}$, we obtain

$$P_2(x)^2 = x^4 - \frac{2}{5}x^2 + \frac{1}{25}.$$

The recurrence gives $b_2 = \frac{18}{35}$ and $b_3 = \frac{20}{21}$, hence $P_4(x) = x^4 - \frac{5}{3}x^2 + \frac{4}{21}$. Substitution and comparison of coefficients yield the identity from the original poster

$$\boxed{P_2(x)^2 = P_4(x) + \frac{19}{15}P_2(x) + \frac{18}{175}P_0(x).}$$

The coefficient $19/15$ multiplying P_2 will reappear as the value at a particular lattice point. This example connects a concrete polynomial calculation with the spatial organization of the coefficients.

3 The Three-Dimensional Coefficient Lattice

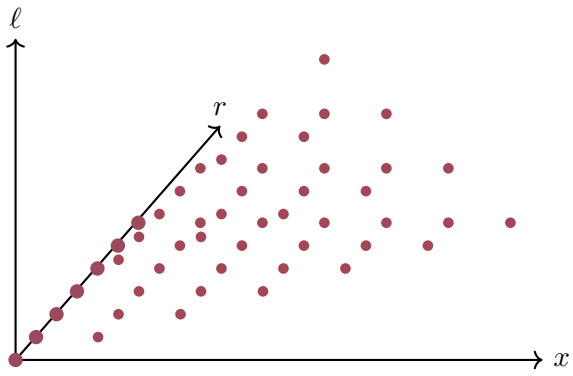
Each index pair (m, n) produces several coefficients through the product expansion. Their arrangement is determined not by their numerical values but by the three indices describing each expansion. Taking the total degree $r = m + n$, the position $x = m$, and the summation index $\ell = j$ gives the assignment

$$H_{r,x,\ell} = F_\ell(x, r - x), \qquad r \geq 0, \quad 0 \leq x \leq r, \quad 0 \leq \ell \leq \min(x, r - x).$$

For fixed r, the admissible pairs (x, ℓ) form a bounded point set symmetric about its center. Each lattice point carries its associated coefficient value; the lattice itself initially describes the admissible indices. The number of points in layer r is

$$N_r = \left\lfloor \frac{(r+2)^2}{4} \right\rfloor.$$

The first layers contain $N_0 = 1$, $N_1 = 2$, $N_2 = 4$, $N_3 = 6$, and $N_4 = 9$ points. For example, Section 2 gives $H_{4,2,1} = F_1(2, 2) = 19/15$, a specific coefficient value in layer $r = 4$.



4 Reflection Symmetry and Algebraic Identities

The shape of the lattice is constrained by its indices. Its internal symmetries arise from polynomial multiplication itself. Since the factors commute, $P_m P_n = P_n P_m$. Comparing the unique product expansions gives

$$F_j(m, n) = F_j(n, m).$$

In lattice coordinates, this becomes the reflection relation

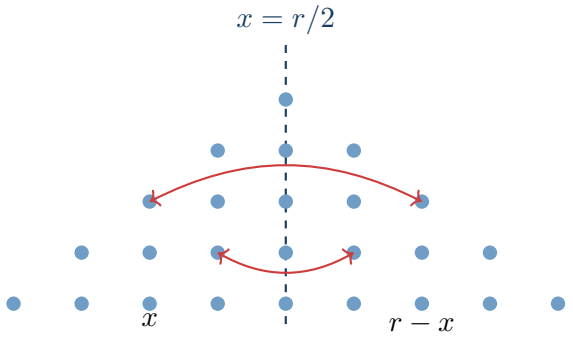
$$H_{r,x,\ell} = H_{r,r-x,\ell}.$$

Each layer is therefore symmetric under reflection about $x = r/2$.

Reflection pairs lattice points with equal coefficient values. The recurrence also provides relations between neighboring product expansions. For $m, n \geq 1$, one obtains

$$\begin{aligned} &F_j(m+1, n) + b_m F_{j-1}(m-1, n) \\ &= F_j(m, n+1) + b_n F_{j-1}(m, n-1). \end{aligned}$$

Coefficients with inadmissible indices are taken to be zero. The identity follows by expanding xP_mP_n once using the recurrence for P_m and once using that for P_n, then comparing coefficients.



5 Local Geometry Through Cross-Ratios

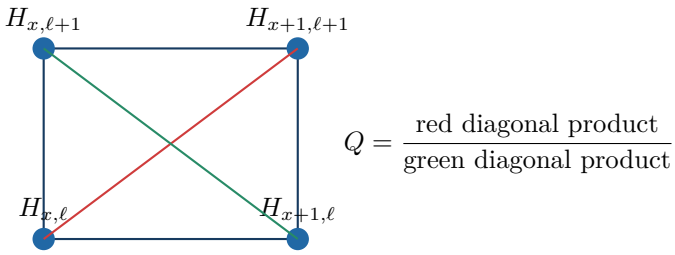
Having examined individual lattice points and their reflections, we now consider four neighboring coefficients together. They form the corners of a lattice cell. Each diagonal connects two opposite values, and the ratio of the products along these diagonals defines a local comparison quantity:

$$Q_{r,x,\ell} = \frac{H_{r,x,\ell} H_{r,x+1,\ell+1}}{H_{r,x+1,\ell} H_{r,x,\ell+1}} > 0.$$

All four points must be admissible. To define the following real logarithmic quantity, the coefficients must also be positive. If the two diagonal products agree, then $Q = 1$; otherwise, the ratio indicates which product is larger. The logarithm

$$K_{r,x,\ell} = \log Q_{r,x,\ell}$$

is referred to here as discrete curvature. When $Q = 1$, we have $K = 0$; when $Q > 1$ or $0 < Q < 1$, K is positive or negative, respectively. This quantity captures the local difference between the two diagonal products.



5.1 Reflection of Curvature

Reflecting a cell interchanges the two diagonal products. Consequently, Q is replaced by its reciprocal, and the logarithm satisfies

$$K_{r,r-x-1,\ell} = -K_{r,x,\ell}.$$

Thus, discrete curvature is antisymmetric between reflected four-point configurations.

6 Associativity as a Global Consistency Condition

The lattice coefficients satisfy more than local relations. They belong to a single multiplication structure in which products of three polynomials are also uniquely defined. Associativity ensures that either way of grouping the factors gives the same result:

$$(P_a P_b)P_c = P_a(P_b P_c).$$

Expanding both sides in the polynomial basis, the coefficient of each P_d must agree. This yields identities between different sums of linearization coefficients.

An explicit form follows by comparing the coefficients of the basis polynomial P_d. Writing $C^q_{m,n}$ for the coefficient of P_q in $P_m P_n$, we obtain

$$\sum_q C^q_{a,b} C^d_{q,c} = \sum_q C^q_{b,c} C^d_{a,q}.$$

Each sum runs over the finitely many admissible degrees. The notation $C^q_{m,n}$ avoids cumbersome parity restrictions; $C^{m+n-2j}_{m,n} = F_j(m,n)$.

The identity states that the coefficient of P_d is unchanged whether P_a and P_b are multiplied first or P_b and P_c are multiplied first. Both sides describe the same polynomial product $P_a P_b P_c$ in the same basis. This identity links different coefficient expansions through the same multiplication structure.

7 Asymptotic Geometry of the Index Set

As r increases, each lattice layer contains more points. To reveal their common geometric shape, the indices are divided by the layer size. This places all layers in the same coordinate region. The normalized coordinates are

$$u = \frac{x}{r}, \qquad v = \frac{\ell}{r} \qquad (r > 0)$$

The inequalities $0 \le x \le r$ and $0 \le \ell \le \min(x, r - x)$ then become

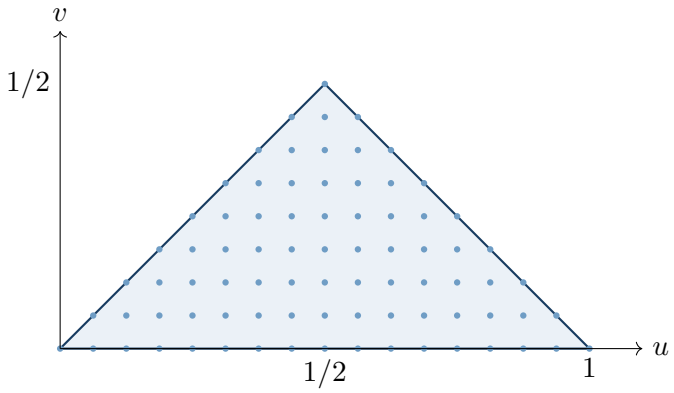
$$0 \le u \le 1, \qquad 0 \le v \le \min(u, 1 - u).$$

The admissible normalized points therefore lie in a triangle with vertices $(0, 0)$, $(1/2, 1/2)$, and $(1, 0)$. As r increases, the points become dense in this region. Its area is

$$\int_0^{1/2} u \, du + \int_{1/2}^{1} (1 - u) \, du = \frac{1}{4}.$$

The point count is consistent with this limiting geometry:

$$N_r = \left\lfloor \frac{(r+2)^2}{4} \right\rfloor, \qquad \lim_{r \to \infty} \frac{N_r}{r^2} = \frac{1}{4}.$$



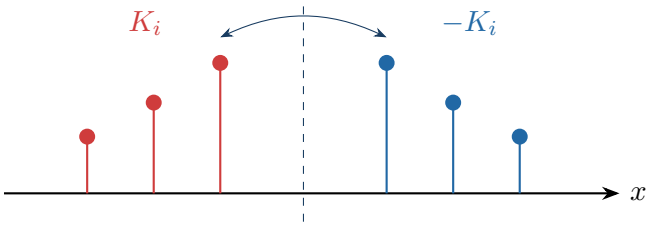
The triangular region gives the limiting geometry of the normalized index lattice.

8 Global Balance of Discrete Curvature

Reflection pairs local curvature values with opposite signs. This gives an exact relation for entire collections of cells: each contribution is canceled by that of its reflected counterpart. The local relation is $K_{r,r-x-1,\ell} = -K_{r,x,\ell}$. Let C_r be a set of admissible four-point configurations invariant under $(x, \ell) \mapsto (r - x - 1, \ell)$, with all cross-ratios positive and defined. The contributions can then be grouped into reflected pairs.

$$\sum_{(x,\ell)\in C_r} K_{r,x,\ell} = 0, \qquad \prod_{(x,\ell)\in C_r} Q_{r,x,\ell} = 1.$$

The second identity follows from the first by exponentiation, since $K = \log Q$. The cancellation works because reflected contributions have opposite signs. Configurations fixed by reflection, lying on the symmetry axis, must therefore satisfy $K = 0$.



The balance holds exactly on every finite layer for reflection-invariant sets of admissible cells.